Planetary Approach Guidance

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Planetary approach guidance, an onboard optical guidance system for a planet-bound spacecraft, functions as the spacecraft approaches the target planet. Such a system should attain an accuracy greater than that obtainable from Earth-based radio guidance. In this paper, a set of orbital parameters especially developed for the problem, including "linearized flight time," is presented. Analytic differential corrections are derived. Next, covariance matrices for orbital parameters including the system biases are given. Of special interest is the manner in which instrument biases can combine with orbital parameters. Finally, a specific mechanization is described, and parametric curves showing accuracy are presented.

I. Introduction

PRESENT injection guidance systems can deliver a spacecraft to Mars or Venus with an accuracy of about 250,000 km (1 σ). Earth-based radio midcourse guidance can effect a considerable improvement, yielding an accuracy of several hundred kilometers (1 σ). The errors in radio guidance are due to 1) errors in performing midcourse maneuvers and 2) orbit-determination errors, which result primarily from errors in physical constants such as the ephemeris of the target planet and the astronomical unit.

If more accuracy is required, as would be the case in landing near a specific point, measurements relating the spacecraft directly to the planet are necessary. In this paper, a self-contained optical system is analyzed for this purpose. As we approach the planet, measurements are made of the angles between the target planet and other suitable bodies, the orbit is then determined, and finally, the necessary maneuvers are computed and executed. Several maneuvers can be made, as needed.

In Sec. II, a set of orbital parameters especially adapted for this problem is described. In Sec. III, an orbit-determination accuracy analysis is carried out using these parameters. A simplified case is employed in order to facilitate the analysis. Of special interest is the way in which biases in the measurement data can combine with the orbital parameters, and a technique is presented for treating this problem. Finally, in Sec. IV, a specific mechanization is described and analyzed. For this mechanization, it is found that accuracy can be improved over radio guidance, although not as markedly as might have been expected, and that the accuracy is limited by a bias in the measurement system that locates the center of the planet.

II. Development of Orbital Parameters

In order to obtain an analytic formulation that will facilitate physical insight into this problem, a conic or two-body approximation will be employed. This approximation agrees well with the true motion out to Egorov's "Sphere of Action,"[1] which for Mars and Venus is about 500,000 km. The differential corrections, being derivatives, agree somewhat further; the conic differential corrections agree with the exact ones to within a few per cent out to 4 to 5 $\times$ 10^6 km from Mars or Venus.[2] Of course, in exact work, precise differential corrections computed from an N-body simulation must be used.

For orbit determination, the partial derivatives of observables with respect to orbit parameters are required. In selecting a set of orbit parameters, we could choose the classical elements, or position and velocity at some epoch, or any other convenient set. Since, in approach guidance, we are especially interested in the amount by which the spacecraft misses the planet, a set of orbital parameters that includes the miss parameter would have advantages. In Ref. 3, the miss or impact parameter $\bar{B}$ is defined as the vector directed from the center of the planet perpendicular to the incoming asymptote of the hyperbola. It is resolved into components $\bar{B}\cdot\bar{R}$ and $\bar{B}\cdot\bar{T}$, where $\bar{R}$ and $\bar{T}$ are unit vectors perpendicular to $\bar{S}$, a unit vector along the incoming asymptote; $\bar{T}$ is in a reference plane such as the ecliptic, and $\bar{R}\bar{S}\bar{T}$ form a right-hand system. Now, define a coordinate system $XYZ(\bar{\imath}\bar{\jmath}\bar{k})$ centered at the planet and such that $\bar{k} = -\bar{S}_s$, $\bar{\imath} = \bar{T}_s$, and $\bar{\jmath} = -\bar{R}_s$, where the subscript s denotes the values on the standard or reference trajectory. Then, the first four of our six orbit parameters q_j, $j = 1, 2, \ldots 6$, are defined as

$$
\begin{aligned}
q_1 &= \bar{B}\cdot\bar{\imath} = \bar{B}\cdot\bar{T}_s\\
q_2 &= -\bar{S}\cdot\bar{\imath} = -\bar{S}\cdot\bar{T}_s\\
q_3 &= \bar{B}\cdot\bar{\jmath} = -\bar{B}\cdot\bar{R}_s\\
q_4 &= -\bar{S}\cdot\bar{\jmath} = \bar{S}\cdot\bar{R}_s
\end{aligned}
\tag{1}
$$

We use $\bar{R}_s$ and $\bar{T}_s$ instead of $\bar{R}$ and $\bar{T}$ in order to prevent the XYZ reference coordinate system from changing when the trajectory is perturbed. The parameters q_1 and q_3 represent miss distances, and q_2 and q_4 represent the angular orientation of the asymptote; q_5 and q_6 remain to be defined.

Since $\bar{B}\cdot\bar{S} = 0$, we have

$$
\begin{aligned}
\bar{S} &= (S_x, S_y, S_z) = [-q_2, -q_4, -(1 - q_2^2 - q_4^2)^{1/2}]\\
\bar{B} &= (B_x, B_y, B_z) = \left(q_1, q_3, -\frac{q_1q_2 + q_3q_4}{(1 - q_2^2 - q_4^2)^{1/2}}\right)
\end{aligned}
\tag{2}
$$

We shall be interested in the properties of hyperbolas, and formulas are presented below for reference. Derivations are not given, but may be found in the literature (e.g., Ref. 4).

From Fig. 1,

$$
\begin{aligned}
c^2 &= A^2 + b^2 \qquad & e &= c/A\\
\cos\phi &= A/c = 1/e \qquad & b &= |\bar{B}|\\
p &= b^2/A = A(e^2 - 1)
\end{aligned}
\tag{3}
$$

Note that $A > 0$. If C_1 and C_3 are, respectively, angular momentum and vis viva, then $b = C_1C_3^{-1/2}$ and $A = \mu/C_3$,

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where μ is GM for the planet. The vectors shown in Fig. 1 are all unit vectors except $\bar{B}$.

From Fig. 1,

$$\begin{aligned} \bar{P} &= \bar{S}(A/c) + (\bar{B}/b)(b/c) \\ \bar{Q} &= \bar{S}(b/c) - (\bar{B}/b)(A/c) \end{aligned} \tag{4}$$

Now, let F be a quantity that is analogous to the eccentric anomaly for ellipses. Reference 4 gives

$$\begin{aligned} A + r &= Ae \cosh F \\ \nu(t - T) &= -F + e \sinh F \end{aligned} \tag{5}$$

where r is the distance from the focus, T is the time of periapsis passage, and ν, the mean motion, is given by $\nu = \mu^{1/2}A^{-3/2}$. We are generally interested in motion prior to periapsis, and hence shall use, in Fig. 1, the segment of the hyperbola in the lower left-hand quadrant. Thus, F will be negative.

For simplicity, we shall often use

$$\begin{aligned} \cosh F &= f \\ \sinh F &= -(f^2 - 1)^{1/2} \end{aligned} \tag{6}$$

Note that Eq. (6) may only be used for $F \leq 0$.

From Ref. 5,

$$\bar{r} = -A\bar{P}(\cosh F - e) + b\bar{Q} \sinh F \tag{7}$$

Using Eq. (4) gives

$$\bar{r} = \bar{S}\Sigma + \bar{B}\beta \tag{8}$$

where

$$\begin{aligned} \Sigma &= (-Af/e) - (b^2/Ae)(f^2 - 1)^{1/2} + A \\ \beta &= 1 - (1/e)[f - (f^2 - 1)^{1/2}] \end{aligned}$$

We now return to the definition of q_5 and q_6. Observe that (q_1, q_2, q_3, q_4) define the asymptote. (Recall that four numbers are required to define a line in three dimensions.) It remains, then, to prescribe the motion along the asymptote. Time of periapsis and energy would appear to be a logical choice. Consider first the time of periapsis. From Eq. (5), we have, for F large and negative,

$$\begin{aligned} A + r &\cong (Ae/2)\, e^{-F} \\ \nu(t - T) &\cong -F - (e/2)\, e^{-F} \end{aligned} \tag{9}$$

Substituting (recall that e is eccentricity),

$$\nu(t - T) = [-(A + r)/A] + \log 2 + \log(A + r) - \log A - \log e \tag{10}$$

Now, $A = \mu/C_3$, and hence A depends only on the energy. But $e = (1 + b^2/A^2)^{1/2}$, and thus the eccentricity depends on the miss parameter. Of course, we expect that, for a given energy and initial distance, a trajectory that hits the center of the planet would arrive at periapsis sooner than a trajectory that misses; the time difference is given by $\Delta T = (\log e)/\nu$. (Note that $t - T$ is negative in the regions of interest.) Since the dependence of ΔT on b is second order, the use of T, the time of periapsis, as an orbit parameter may generate undesirable nonlinearities. Also, the use of T would cause bothersome coupling between lateral differential corrections and those along the asymptote. These difficulties can be circumvented by adjusting the time of periapsis by an amount ΔT. Let

$$\begin{aligned} q_5 &= T_e = T - (\log e)/\nu \\ \nu(t - q_5) &= \log e - F + e \sinh F \end{aligned} \tag{11}$$

T_e is called "linearized flight time." Finally, we set $q_6 = C_3$, where C_3, the vis viva, is twice the energy and equals the square of the asymptotic velocity.

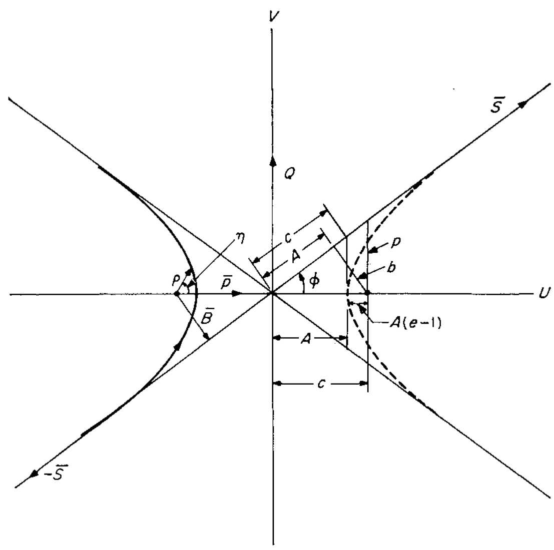

Fig. 1 Geometry of hyperbola.

The differential corrections relating observables to orbital parameters are given in the Appendix.

III. Orbit Determination for Approach Guidance

The Gauss-Markov[6] formulation will be used for orbit determination. In order to illustrate the basic character of the process, a simple case will first be treated using the following assumptions:

1) The nominal path hits the center of the planet, and in the region of interest, the spacecraft is not near the planet, i.e., the path is almost a straight line.

2) At each of i points, where $i = 1, 2, \ldots N$, three measurements are made: θ_i^1, the angle between the center of the planet and a star in the direction of the $+X$ axis; θ_i^2, the angle between the center of the planet and a star in the $+Y$ direction; and ψ_i, the angular diameter of the planet.

3) The noise on the measurements is random and stationary; no cross- or autocorrelations are present.

4) No physical constant uncertainties, data biases, etc., are present.

5) No a priori data are available.

Later, we shall weaken these assumptions.

For a hitting trajectory, we can use the position derivatives of Eq. (A8). Also, since we assume that the spacecraft is not close to the planet, f is large and $\beta \approx 1$. From the Appendix, we have $\bar{U}^1 = (1, 0, 0)$; thus, from Eq. (A10),

$$\frac{\partial\theta_i^1}{\partial\bar{r}} = \left(\frac{\partial\theta_i^1}{\partial x}, \frac{\partial\theta_i^1}{\partial y}, \frac{\partial\theta_i^1}{\partial z}\right) = \left(\frac{1}{r_i}, 0, 0\right) \tag{12}$$

where r_i is the nominal value of r at the ith measurement. Also, $\partial\theta_i^2/\partial\bar{r} = (0, 1/r_i, 0)$ and $\partial\psi_i/\partial\bar{r} = (0, 0, -d/r_i^2)$.

Now, let $\boldsymbol{\delta\Theta}^1$ be the column matrix with elements $\delta\theta_i^1$, $i = 1, 2, \ldots N$.‡ Similarly, define the column matrices $\boldsymbol{\delta\Theta}^2$ and $\boldsymbol{\delta\psi}$. Finally, let $\boldsymbol{\delta\Theta}$ be a column matrix with (partitioned) elements $\boldsymbol{\delta\Theta}^1$, $\boldsymbol{\delta\Theta}^2$, and $\boldsymbol{\delta\psi}$. We may then write $\boldsymbol{\delta\Theta} = \mathbf{A}\boldsymbol{\delta}\mathbf{q}$ where $\mathbf{A}$ is the $3N \times 6$ matrix of derivatives of observables (angles) with respect to the orbit parameters q_j, and $\boldsymbol{\delta}\mathbf{q}$ is the column matrix of perturbations with elements δq_j, $j = 1, 2, \ldots 6$.

By the use of Eqs. (A9) and (A11), we have

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \boldsymbol{\phi} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \mathbf{A}_2 & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \boldsymbol{\phi} & \mathbf{A}_3 \end{pmatrix} \tag{13}$$

‡ Matrices are indicated by boldface type.

where $\boldsymbol{\phi}$ is the $N \times 2$ zero matrix, $\mathbf{A}_1$ is $N \times 2$ having elements $1/r_i$ in the first column and 1's in the second column, and $\mathbf{A}_2 = \mathbf{A}_1$. Similarly, $\mathbf{A}_3$ is $N \times 2$ having elements $d\dot{r}_i/r_i^2$ in the first column and $1/2r_iq_6$ in the second column.

The normal matrix is $\mathbf{J} = \mathbf{A}'\boldsymbol{\lambda}^{-1}\mathbf{A}$, where the prime denotes transpose and $\boldsymbol{\lambda}$ is the covariance of the noise on the observations. Since we have assumed stationary, uncorrelated noise, we may write

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1^2\mathbf{I} & \boldsymbol{\phi} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \lambda_2^2\mathbf{I} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \boldsymbol{\phi} & \lambda_3^2\mathbf{I} \end{pmatrix} \tag{14}$$

where $\boldsymbol{\lambda}$ is $3N \times 3N$, $\mathbf{I}$ is the $N \times N$ identity matrix, and $\boldsymbol{\phi}$ is the $N \times N$ zero matrix. The mean square noise on the observations of $\delta\theta^1$, $\delta\theta^2$, and $\delta\psi$ are, respectively, λ_1^2, λ_2^2, and λ_3^2.

Finally,

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \boldsymbol{\phi} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \mathbf{J}_2 & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \boldsymbol{\phi} & \mathbf{J}_3 \end{pmatrix} \tag{15}$$

where $\mathbf{J}_1 = (1/\lambda_1^2)\mathbf{A}_1'\mathbf{A}_1$, and $\mathbf{J}_2$ and $\mathbf{J}_3$ are similarly defined.

The covariance of $\boldsymbol{\delta}\mathbf{q}^*$, the estimator of $\boldsymbol{\delta}\mathbf{q}$, is the inverse of the normal matrix. If $\boldsymbol{\Gamma}$ is the covariance of the estimator, then $\boldsymbol{\Gamma} = \mathbf{J}^{-1}$. Note that to obtain $\mathbf{J}^{-1}$ from Eq. (15), we simply invert each partitioned element, letting $\boldsymbol{\phi}^{-1} = \boldsymbol{\phi}$.

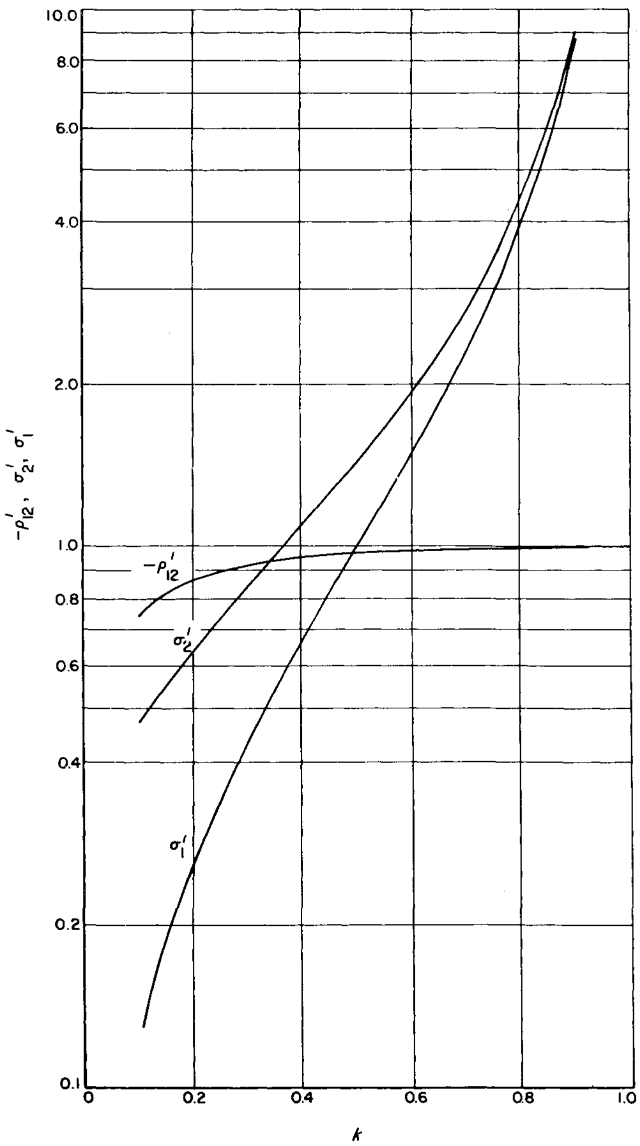

Fig. 2 Accuracy, simplified model, no a priori data.

Observe that the determination of q_1 and q_2 depends only on θ^1, and no correlation exists between the determination of (q_1, q_2) and any of the other four parameters. Next, write

$$\mathbf{J}_1^{-1} = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{16}$$

where we read σ_1^2 and σ_2^2, respectively, as the variance of the estimation of q_1 and q_2, with ρ_{12} being the correlation coefficient. From the preceding, we have

$$\left.\begin{aligned} \sigma_1^2 &= N(\lambda_1^2/D) \\ \sigma_2^2 &= \left(\sum_{i=1}^{N} \frac{1}{r_i^2}\right)\frac{\lambda_1^2}{D} \\ \rho_{12}\sigma_1\sigma_2 &= -\;[\Sigma(1/r_i)](\lambda_1^2/D) \end{aligned}\right\} \tag{17}$$

where

$$D = N\sum_i \frac{1}{r_i^2} - \left(\sum_i \frac{1}{r_i}\right)^2 = \sum_{i=1}^{N}\sum_{j=1}^{i}\left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 \tag{18}$$

and in the double sum of Eq. (18), each pair (r_i, r_j) is taken once and only once.

From Eqs. (17) and (18), we see, as expected, that a single measurement ($N = 1$) is inadequate to determine the orbit. Also, to minimize σ_1^2, we would like to choose pairs of points (r_i, r_j) so as to maximize $(1/r_i - 1/r_j)^2$. Indeed, to maximize D, we should take half of the measurements at the farthest distance and the other half at the closest. In practice, this is probably not feasible, and, at any rate, the noise on two closely spaced observations is not likely to be uncorrelated.

Figure 2 illustrates the behavior of σ_1, σ_2, and ρ_{12}. In the figure, it is assumed that 10 measurements are made ($N = 10$), and that they are equally spaced between r_1 and r_{10}. Letting $r_1 = kr_{10}$, Fig. 2 gives plots of ρ_{12}, $\sigma_1' = \sigma_1/\lambda_1 r_{10}$, and $\sigma_2' = \sigma_2/\lambda_1 r_{10}$. Note that ρ_{12} approaches -1 as k approaches unity.

An a priori estimate of the orbit parameters $\mathbf{q}$ may be available from, say, radio tracking. If this estimate is $\mathbf{q}_0^*$ with covariance $\boldsymbol{\Lambda}_0$, then $\boldsymbol{\Gamma} = (\boldsymbol{\Lambda}_0^{-1} + \mathbf{J})^{-1}$, where $\boldsymbol{\Gamma}$ is the covariance of the estimator which combines both a priori and tracking data, and $\mathbf{J}$ is the normal matrix of the tracking data.

Suppose for simplicity that $\boldsymbol{\Lambda}_0$ is diagonal. Let the first two terms along the diagonal be δ_1^2 and δ_2^2; thus, δ_1 represents the uncertainty in our a priori knowledge of q_1, etc. If $\boldsymbol{\Gamma}_1$ is the upper left-hand 2×2 matrix in $\boldsymbol{\Gamma}$, then we may write

$$\boldsymbol{\Gamma}_1 = \begin{pmatrix} \gamma_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix} \tag{19}$$

Proceeding as before, using the data-gathering pattern and noise model of the preceding example,

$$\gamma_1^2 = \frac{\lambda_1^2\left(N + \dfrac{\lambda_1^2}{\delta_2^2}\right)}{\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{i}\left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + \frac{\lambda_1^4}{\delta_1^2\delta_2^2} + \frac{\lambda_1^2}{\delta_2^2}\sum_{i=1}^{N}\frac{1}{r_i^2} + \frac{\lambda_1^2 N}{\delta_1^2}} \tag{20}$$

where we may regard γ_1 as our a posteriori uncertainty in q_1. In Fig. 3, γ_1' is plotted, where $\gamma_1' = \gamma_1/\lambda_1 r_{10}$. The data-gathering pattern of Fig. 2 was again used. The four curves are described as follows: Curve 1: $\delta_1 = \delta_2 = \infty$ (no a priori data; this is σ_1' of Fig. 2); Curve 2: $\delta_2 = \infty$, $\delta_1 = \lambda_1 r_0$ (a priori data on q_1 only); Curve 3: $\delta_1 = \infty$, $\delta_2 = \lambda_1$ (a priori data on q_2 only); Curve 4: $\delta_1 = \lambda_1 r_0$, $\delta_2 = \lambda_1$ (a priori data on q_1 and q_2). As expected, additional information diminishes the variance of the estimate.

Biases may be present in the measurement equipment. A bias is defined as an error affecting the measurements and fixed over a given experiment. Thus, the biases may be treated simply as additional orbit parameters.

Care must be exercised, however, if the biases affect the measurements in the same manner as the orbit parameters. For example, in the case just treated, we have (for straight-line motion) $\theta^1 - \pi/2 = q_1/r + q_2$. If a bias q_7 is present in the apparatus which measures θ_1, we observe, instead, $q_1/r + q_2 + q_7$. Thus, q_2 and q_7 occur in linear combination and cannot be dissociated by observing θ^1. If q_7 is included as an orbit parameter, **J** will be singular. Of course, a priori data afford a conditioning which may permit $\mathbf{\Gamma}$ to exist.

For almost straight-line motion, angle biases corrupt q_2 and q_4; similarly, biases in locating the center of the planet corrupt q_1 and q_3, assuming that this bias is some fraction of the diameter of the planet (rather than a fixed angle), as would occur when the sensor that is attempting to locate the center of the planet selects some fixed point other than the center (of the visible disk) and judges this fixed point to be the center.

Even without a priori data, we can estimate q_1 and q_3 when angle biases are present; similarly, we can estimate q_2 and q_4 when planet sensor biases are present. However, when both types of bias are present, neither (q_1, q_3) nor (q_2, q_4) can be estimated without a priori data.§ (Note that in the near vicinity of the planet, the curvature of the trajectory will permit the orbit parameters to be distinguished from the biases.)

We may generalize by writing

$$\delta\mathbf{\Theta} = \mathbf{A}_e\delta\mathbf{q}_e = (\mathbf{AT})\begin{pmatrix}\delta\mathbf{q}\\ \delta\mathbf{q}_b\end{pmatrix} = \mathbf{A}\delta\mathbf{q} + \mathbf{T}\delta\mathbf{q}_b \qquad (21)$$

where $\mathbf{A}_e = (\mathbf{AT})$, $\delta\mathbf{\Theta}$ is the measurements, and $\delta\mathbf{q}_e$ is the extended matrix of generalized orbit parameters consisting of the original parameters $\delta\mathbf{q}$ and biases $\delta\mathbf{q}_b$.

Suppose now that in Eq. (21) a linear combination of $\delta\mathbf{q}$ and $\delta\mathbf{q}_b$ exists which our measurements (characterized by $\mathbf{A}_e$) do not permit us to separate. Then, we should be able to write $\delta\mathbf{\Theta} = \mathbf{A}(\delta\mathbf{q} + \mathbf{C}\delta\mathbf{q}_b)$; hence, $\mathbf{T} = \mathbf{AC}$, where **C** describes the linear relationships between biases and orbit parameters.

Note that if we define $\delta\mathbf{Q}$ such that $\delta\mathbf{Q} = (\mathbf{IC})\delta\mathbf{q}_e$, where **I** is the identity matrix, we have $\delta\mathbf{\Theta} = \mathbf{A}\delta\mathbf{Q}$. Therefore, if biases are present but are unknown or ignored, it is $\delta\mathbf{Q}$ that we estimate rather than $\delta\mathbf{q}$.

Finally, we shall develop an expression which displays the effects of the linearly dependent biases on the determination of the orbit parameters $\delta\mathbf{q}$, when a priori data are available. Let

$$\mathbf{\Lambda}_e = \begin{pmatrix}\mathbf{\Lambda}_0 & \mathbf{\phi}\\ \mathbf{\phi}' & \mathbf{B}\end{pmatrix} \qquad (22)$$

where as before, $\mathbf{\Lambda}_0$ is the covariance of our a priori knowledge of $\delta\mathbf{q}$, **B** is the covariance of our a priori knowledge of the biases $\delta\mathbf{q}_b$, and $\mathbf{\phi}$ is a zero matrix. Letting $\mathbf{J}_e = \mathbf{A}_e'\mathbf{\lambda}^{-1}\mathbf{A}_e$, $\mathbf{\Gamma}_e^{-1} = \mathbf{\Lambda}_e^{-1} + \mathbf{J}_e$. By the use of the matrix identity given in Ref. 7, we can show that $\mathbf{\Gamma}_q$, the a posteriori covariance of $\delta\mathbf{q}$, i.e., the upper left-hand portion of $\mathbf{\Gamma}_e$, is given by

$$\mathbf{\Gamma}_q^{-1} = \mathbf{\Lambda}_0^{-1} + (\mathbf{J}^{-1} + \mathbf{CBC}')^{-1} \qquad (23)$$

where, as before, $\mathbf{J} = \mathbf{A}'\mathbf{\lambda}^{-1}\mathbf{A}$.

Finally, the Schmidt-Kalman form is given. Suppose that prior to the $(i + 1)$ measurement point we have an estimator of the orbital parameters (including biases) with covariance $\mathbf{\Gamma}_i$. Suppose further that at the $(i + 1)$ point, we make measurement(s) having partial derivatives A_{i+1} and (measurement) covariance $\mathbf{\lambda}_{i+1}$. Then, if the noise on the $(i + 1)$ measurements is not correlated with previous data, we have, from Ref. 6, $\mathbf{\Gamma}_{i+1}^{-1} = \mathbf{\Gamma}_i^{-1} + \mathbf{A}_{i+1}'\mathbf{\lambda}_{i+1}^{-1}\mathbf{A}_{i+1}$. Using again the identity from Ref. 7, we obtain

$$\mathbf{\Gamma}_{i+1} = \mathbf{\Gamma}_i[\mathbf{I} - \mathbf{A}_{i+1}'(\mathbf{\lambda}_{i+1} + \mathbf{A}_{i+1}\mathbf{\Gamma}_i\mathbf{A}_{i+1}')^{-1}\mathbf{A}_{i+1}\mathbf{\Gamma}_i] \qquad (24)$$

The estimator may be obtained similarly. This form, which is due to Schmidt[8] and Kalman,[9] is convenient when estimates and/or covariances are desired at each measurement point.

IV. Results for a Specific Mechanization

It is assumed that the spacecraft is oriented by a two-axis sun sensor that points the roll axis toward the sun, and by a single-axis star sensor that acquires the star Canopus and holds the spacecraft fixed in roll. An optical device that measures the angular diameter of the planet and finds the center of the visible disk completes the measurement system.¶ The angle between the sun and the planet, called the cone

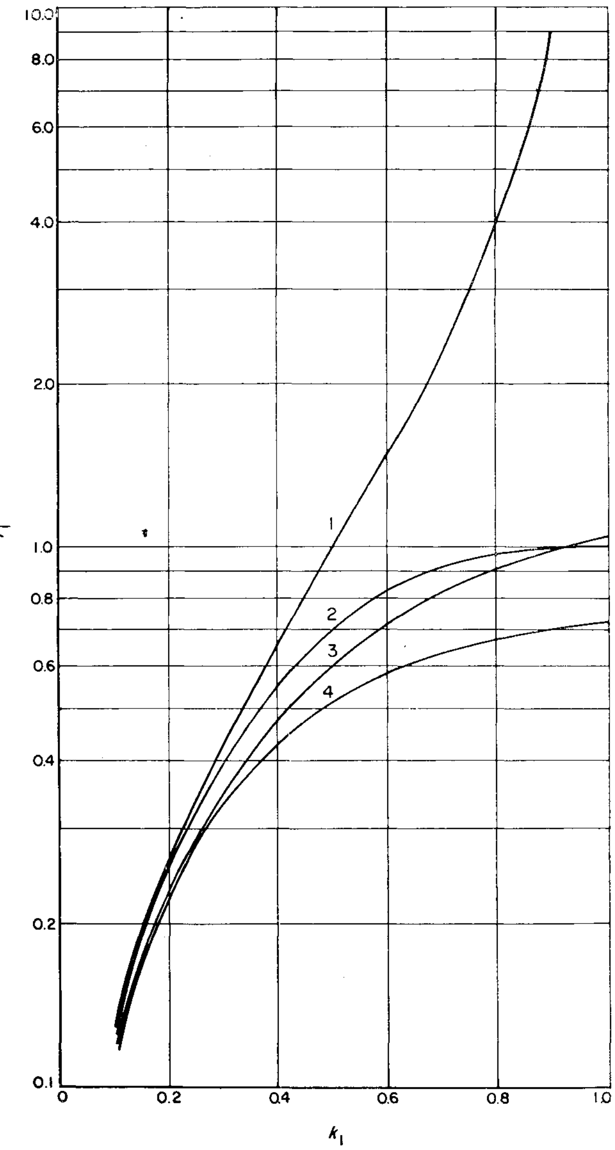

Fig. 3 Accuracy, simplified model, with a priori data.

§ The authors are indebted to a reviewer, Gerald L. Smith of NASA Ames Research Center, for pointing out that penrose's generalized inverse (Refs. 12–14) permits a "best approximate estimate" to be made in this case even though J is singular.

¶ The sun sensor and the star tracker have been mechanized and used. The Celestial Sensors Group of the Guidance and Control Division at Jet Propulsion Laboratory is designing an instrument to be used to locate the center of a partially illuminated disk as well as to measure its angular diameter.

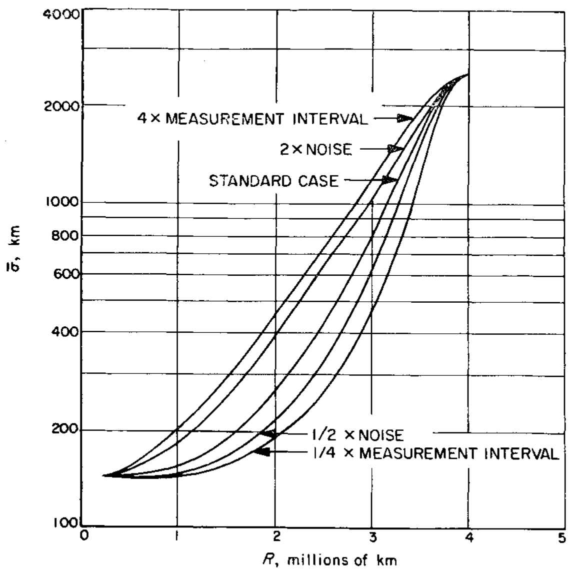

Fig. 4 Accuracy, hypothetical system, variations in noise and measurement interval.

angle, is taken as one measurement type, and the angle between the sun-spacecraft-planet and sun-spacecraft-Canopus planes, called the clock angle, as the second. (Notice that cone and clock angles are analogous to colatitude and longitude, respectively, on a celestial sphere with the sun at the pole.) The angular diameter of the planet is the third measurement type. The errors in cone and clock angles may be correlated at each measurement point, since each uses the planet-center finder as well as common mountings and components. These correlations can be handled easily by the techniques described earlier if one assumes that there is no correlation between adjacent measurement points.

The system described previously has been simulated with an IBM 7094 program using the differential correction formulas of the Appendix. Both Gauss-Markov and Schmidt-Kalman forms were programed; the numerical results obtained were identical providing that the a priori covariance matrix was adequate to prevent ill-conditioning of the a posteriori covariance matrix.

A typical trajectory, launched December 22, 1966, and reaching closest approach to Mars on July 11, 1967, was used to illustrate the results of this onboard system. The encounter geometry is such that the orbital parameters are $\mathbf{q}' = (14{,}000 \text{ km};\ 0;\ -5000 \text{ km};\ 0;\ 0;\ 40.3 \text{ km}^2/\text{sec}^2)$.

For the purposes of this study, a relatively pessimistic a priori covariance matrix was generated by mapping the results of a hypothetical post-midcourse orbit determination to the parameters used to describe the approach trajectory. It was assumed that the orbit determination had standard deviations associated with velocity components of 0.003 m/sec along the velocity vector and 0.03 m/sec perpendicular to it and with position components of 30 km in each direction, with epoch at midcourse time.

A standard measurement pattern is defined to be 100 measurements spaced at equal intervals, the first at 4×10^6 km and the last at 0.2×10^6 km. The cone- and clock-angle measurements were assumed to have gaussian noise with standard deviations of 0.1 and 0.16 mrad, respectively, and a correlation coefficient of 0.01. The noise on the planetary angular-diameter measurements was assumed gaussian with a standard deviation of 1.0 mrad.** The biases, though constant for any given flight, are zero mean gaussian variables for an ensemble of flights. The cone- and clock-angle biases were assumed to have standard deviations of 1.0 and 1.9 mrad, respectively, with a correlation coefficient of 0.05. The planet-center finder bias was assumed to have a standard deviation of 100 km in two orthogonal directions.

The quality of the orbit determination is described by the covariance matrix of the estimate. Table 1 gives the complete covariance of the estimator for the standard case. The first six parameters correspond to those previously described; the other four are defined in Table 1. Considering that the primary purpose of approach guidance is to pass the planet at some specific position $\sigma = (\sigma_1^2 + \sigma_3^2)^{1/2}$, the root-mean-square of the standard deviations about the estimate of the position coordinates is used as a figure of merit. Figures 4 and 5 show σ as a function of range. Figure 4 shows the effect of noise as well as the effect of varying the measurement density. Notice that increasing the measurement density has exactly the same effect as decreasing the noise. Figure 5 shows the effect of angle bias, planet-center finder bias, and a priori data. Notice that the angle biases degrade the orbit parameters when measurements start, but that after enough data have been accumulated to estimate those biases accurately, they no longer affect the estimate of the orbit parameters. The limiting factor on the orbit-determination accuracy is seen to be the planet-center finder biases. During the approach phase, the partial derivatives of observables with respect to the positions q_1 and q_3 are the same as with respect to these biases, and so their errors are combined.

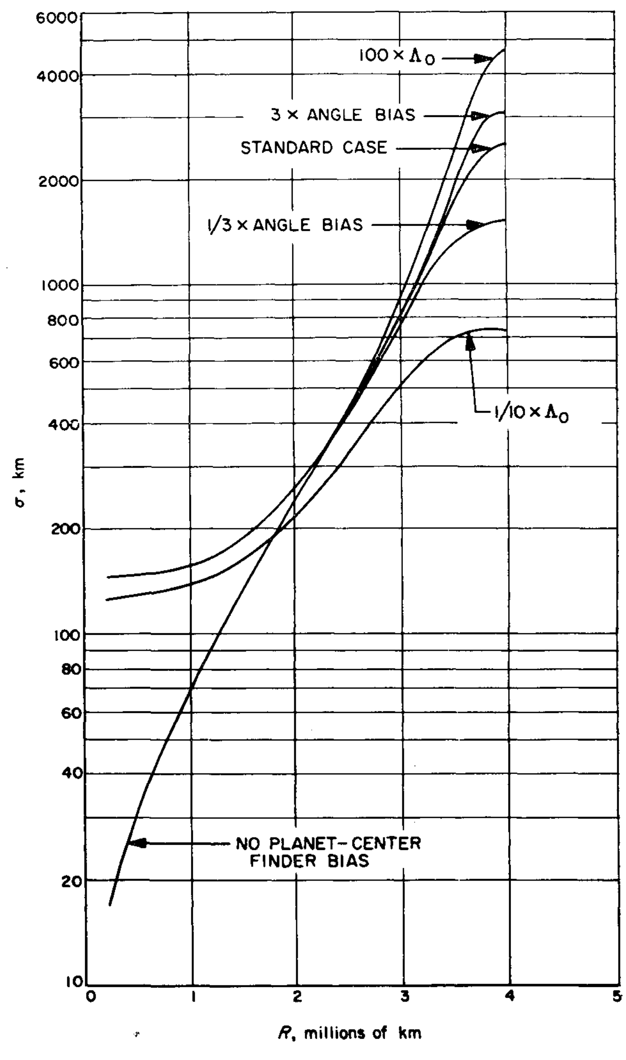

Fig. 5 Accuracy, hypothetical system, variations in bias and initial conditions.

** The approach phase begins when the angular diameter is 0.1°, at which time its measurements are much less precise than the other two angular measurements and do not contribute significantly to the orbit determination.

Additional results that have been obtained include the magnitude, optimal spacing, and execution error effects of approach guidance maneuvers[10] and effects of drifts superimposed on the biases.[11]

Reference 15, which appeared after the writing of this paper, develops a Monte Carlo technique for analyzing planetary approach orbit determination based on noisy measurements. The analysis we present here differs from that of Ref. 15 in that an analytic technique is employed and instrument biases, which appear to control the resultant accuracy, are considered.

Appendix: Differential Corrections

We require positional differential corrections $\partial\bar{r}/\partial q_j$, $j = 1, 2, \ldots, 6$. From Eq. (8), we have

$$\frac{\partial\bar{r}}{\partial q_j} = \bar{S}\frac{\partial\Sigma}{\partial q_j} + \frac{\partial\bar{S}}{\partial q_j}\Sigma + \bar{B}\frac{\partial\beta}{\partial q_j} + \frac{\partial\bar{B}}{\partial q_j}\beta \tag{A1}$$

From Eq. (2), we have

$$\begin{aligned}
&\frac{\partial\bar{S}}{\partial q_j} = 0, 0, 0 \qquad j = 1, 3, 5, 6 \\
&\frac{\partial\bar{S}}{\partial q_2} = -1, 0, \frac{q_2}{g} \qquad \frac{\partial\bar{S}}{\partial q_4} = 0, -1, \frac{q_4}{g} \\
&\frac{\partial\bar{B}}{\partial q_1} = 1, 0, -\frac{q_2}{g} \\
&\frac{\partial\bar{B}}{\partial q_2} = 0, 0, -\frac{1}{g}\left(q_2\frac{k}{g^2} + q_1\right) \\
&\frac{\partial\bar{B}}{\partial q_4} = 0, 0, -\frac{1}{q}\left(q_4\frac{k}{g^2} + q_3\right) \\
&\frac{\partial\bar{B}}{\partial q_3} = 0, 1, -\frac{q_4}{g} \qquad \frac{\partial\bar{B}}{\partial q_5} = \frac{\partial\bar{B}}{\partial q_6} = 0, 0, 0
\end{aligned} \tag{A2}$$

where $g = (1 - q_2^2 - q_4^2)^{1/2}$ and $k = q_1q_2 + q_3q_4$.

To obtain the remaining terms of Eq. (A1), we use Eqs. (8) and (10), recalling that $A = \mu/q_6$, $e = c/A$, and $b^2 = q_1^2 + q_3^2 + k^2/g^2$, from which derivatives $\partial b^2/\partial q_j$ can be obtained. Note that

$$\left.\begin{aligned}
&\frac{\partial c}{\partial q_j} = \frac{1}{2c}\frac{\partial b^2}{\partial q_j} \qquad j = 1, 2, 3, 4 \\
&\frac{\partial c}{\partial q_5} = 0 \qquad \frac{\partial c}{q q_6} = \frac{-\mu^2}{\partial_6{}^3 c}
\end{aligned}\right\} \tag{A3}$$

$$\left.\begin{aligned}
&\frac{\partial e}{\partial q_j} = \frac{1}{2Ac}\frac{\partial b^2}{\partial q_j} \qquad j = 1, 2, 3, 4, 5 \\
&\frac{\partial e}{\partial q_6} = \frac{b^2}{\mu c}
\end{aligned}\right\} \tag{A4}$$

In differentiating Eq. (8), we shall obtain terms $\partial f/\partial q_j$, which can be obtained from Eq. (11), yielding

$$\left.\begin{aligned}
&\frac{\partial F}{\partial q_j} = \frac{1}{2Ace}\,\frac{1 - e(f^2-1)^{1/2}}{1 - ef}\,\frac{\partial b^2}{\partial q_j} \qquad j = 1, 2, 3, 4 \\
&\partial F/\partial q_5 = q_6{}^{3/2}/\mu\,(1 - ef) \\
&\frac{\partial F}{\partial q_6} = \frac{\dfrac{b^2}{\mu c e}\left[1 - e(f^2-1)^{1/2}\right] - \dfrac{3q_6{}^{1/2}}{2\mu}(t - q_5)}{1 - ef}
\end{aligned}\right\} \tag{A5}$$

Substituting gives

$$\left.\begin{aligned}
&\frac{\partial\Sigma}{\partial q_j} = \frac{\partial b^2}{\partial q_j}\left\{\frac{f}{2Ae^3} - \frac{1}{Ae}\left(1 - \frac{b^2}{2A^2e^2}\right)(f^2-1)^{1/2} + \left(\frac{1}{2A^3e^3}\right)\frac{1 - e(f^2-1)^{1/2}}{1 - ef}\cdot[b^2f + A^2(f^2-1)^{1/2}]\right\} \\
&\qquad j = 1, 2, 3, 4 \\
&\frac{\partial\Sigma}{\partial q_5} = \frac{q_6{}^{1/2}}{A^2e}\cdot\frac{1}{1 - ef}[b^2f + A^2(f^2-1)^{1/2}] \\
&\frac{\partial\Sigma}{\partial q_6} = \frac{2Af}{q_6e} - \frac{A}{q_6} - \frac{1}{q_6Ae^3}[A^2f + b^2(f^2-1)^{1/2}] + \frac{1}{Ae}[b^2f + A^2(f^2-1)^{1/2}]\frac{\partial F}{\partial q_6} \\
&\frac{\partial F}{\partial q_6} = \frac{1}{1 - ef}\left\{\frac{b^2}{\mu Ae^2}[1 - e(f^2-1)^{1/2}] - \frac{3}{2q_6}[\log e + \cosh^{-1}f - e(f^2-1)^{1/2}]\right\} \\
&\frac{\partial\beta}{\partial q_j} = \frac{\partial b^2}{\partial q_j}\,\frac{[f - (f^2-1)^{1/2}]}{2A^2e^3}\cdot\left(1 - \frac{1 - e(f^2-1)^{1/2}}{1 - ef}\right) \qquad j = 1, 2, 3, 4 \\
&\frac{\partial\beta}{\partial q_5} = -\frac{q_6{}^{1/2}}{Ae}\cdot\frac{f - (f^2-1)^{1/2}}{1 - ef} \\
&\frac{\partial\beta}{\partial q_6} = \frac{1}{e}[f - (f^2-1)^{1/2}]\left[\frac{1}{q_6}\left(1 - \frac{1}{e^2}\right) - \frac{\partial F}{\partial q_6}\right]
\end{aligned}\right\} \tag{A6}$$

We may also require $\partial r/\partial q_j$, which can be obtained from Eq. (5), and is given by

$$\begin{aligned}
&\frac{\partial r}{\partial q_j} = \frac{\partial b^2}{\partial q_j}\left[\frac{f}{2Ae} - \frac{(f^2-1)^{1/2}}{2Ae}\cdot\frac{1 - e(f^2-1)^{1/2}}{1 - ef}\right] \qquad j = 1, 2, 3, 4 \\
&\frac{\partial r}{\partial q_5} = \frac{\mu e}{q_6{}^{1/2}r}(f^2-1)^{1/2} = -\dot{r} \\
&\frac{\partial r}{\partial q_6} = \frac{A}{q_6}(1 - ef) + \frac{fb^2}{\mu e} - Ae(f^2-1)^{1/2}\frac{\partial F}{\partial q_6}
\end{aligned} \tag{A7}$$

The velocity differential corrections $\partial\dot{\bar{r}}/\partial q_j$ may be obtained by the preceding technique and will not be given here.

In order to gain some insight into the nature of these differential corrections, let us examine the trajectory that passes through the center of the planet. Here we have $b = 0$, $e = 1$, etc. We find

$$\begin{aligned}
&\frac{\partial\bar{r}}{\partial q_1} = \beta, 0, 0 \qquad \frac{\partial\bar{r}}{\partial q_4} = 0, r, 0 \\
&\frac{\partial\bar{r}}{\partial q_2} = r, 0, 0 \qquad \frac{\partial\bar{r}}{\partial q_5} = 0, 0, -\dot{r} \\
&\frac{\partial\bar{r}}{\partial q_3} = 0, \beta, 0 \qquad \frac{\partial\bar{r}}{\partial q_6} = 0, 0, -\frac{\partial\Sigma}{q_6}
\end{aligned} \tag{A8}$$

The presence of zeros, especially in the Z positions of the q_1, q_2, q_3, and q_4 derivatives and in the X and Y positions for the q_5 and q_6 derivatives, indicates that the decoupling has been successful. Note that $\dot{r}$ is negative in the region of interest.

Table 1 Covariance of the estimator

j	jth orbital parameter	Standard deviation
1	q_1	107.3 km
2	q_2	0.0592 mrad
3	q_3	81.0 km
4	q_4	0.0588 mrad
5	q_5	152.6 sec
6	q_6	0.00194 km²/sec²
7	Bias on cone angle	0.0309 mrad
8	Bias on clock angle	0.0546 mrad
9	Center-finder bias in direction of q_1	69.6 km
10	Center-finder bias in direction of q_3	71.4 km

Correlations between orbit parameters ($\rho_{ij} = \rho_{ji}$)

i/j	1	2	3	4	5	6	7	8	9	10
1	1	0.427	−0.246	0.141	−0.130	0.347	0.166	0.254	0.970	0.916
2		1	−0.045	0.478	−0.135	0.240	0.596	0.331	0.238	0.287
3			1	−0.585	0.046	0.339	0.174	−0.351	−0.642	−0.881
4				1	−0.237	0.184	−0.336	0.495	0.419	0.152
5					1	−0.563	−0.208	−0.534	−0.517	−0.592
6		(Symmetric)				1	0.407	0.408	0.287	0.378
7							1	0.119	0.097	0.059
8								1	0.287	−0.055
9									1	0.181
10										1

If we are far away from the planet, such that f is large, a few simplifications occur for the hitting case. Including the velocity differential corrections, we may write: $\boldsymbol{\delta}\mathbf{x} = \mathbf{L}\boldsymbol{\delta}\mathbf{q}$, where $\boldsymbol{\delta}\mathbf{x}$ is a column matrix with elements (δx, $\delta \dot{x}$, δy, $\delta \dot{y}$, δz, $\delta \dot{z}$), $\boldsymbol{\delta}\mathbf{q}$ is a column matrix with elements δq_j, and

$$\mathbf{L} = \begin{pmatrix} \mathbf{M} & \boldsymbol{\phi} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \mathbf{M} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \boldsymbol{\phi} & \mathbf{N} \end{pmatrix} \tag{A9}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & r \\ 0 & \dot{r} \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} -\dot{r} & r/2q_6 \\ \mu/r^2 & 1/2q_6^{1/2} \end{pmatrix}$$

We may easily invert $\mathbf{L}$, and $\mathbf{L}$ and $\mathbf{L}^{-1}$ may be checked by referring to elementary principles.

It remains, finally, to find the derivatives of observables with respect to the q_j. Since the observables, being angles, are functions only of position, our task is easy. Let θ be the angle observed at the spacecraft between the center of the planet and a star. If $\bar{U}^1 = (U_x{}^1, U_y{}^1, U_z{}^1)$ is a unit vector from spacecraft to star and $\bar{R}^1$ a unit vector from spacecraft to planet, then $\cos\theta = \bar{U}^1 \cdot \bar{R}^1$. Now, $\bar{R}^1 = -\bar{r}/r$, where $\bar{r}$ is the position vector from planet to spacecraft in our XYZ system. Differentiating the preceding expression with respect to x, using $\partial r/\partial x = x/r$, and assuming that the star is infinitely far away such that $\bar{U}^1$ is not a function of position, we have

$$\frac{\partial \theta}{\partial x} = \frac{U_x{}^1 + (x/r)\cos\theta}{r \sin\theta} \tag{A10}$$

The derivatives with respect to x and y are obtained similarly. To obtain $\partial\theta/\partial q_j$, we use

$$\frac{\partial \theta}{\partial q_j} = \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial q_j} + \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial q_j} + \frac{\partial \theta}{\partial z}\frac{\partial z}{\partial q_j} \tag{A11}$$

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